The boundary layer on a spherical gas bubble

By D. W. MOORE

Department of Mathematics, Bristol University

(Received 4 August 1962 and in revised form 24 November 1962)

The equations governing the boundary layer on a spherical gas bubble rising steadily through liquid of small viscosity are derived. These equations are linear and are solved in closed form. The boundary layer separates at the rear stagnation point of the bubble to form a thin wake, whose structure is determined. Thus the drag force can be calculated from the momentum defect. The value obtained is $12\pi aU\mu$, where a is the bubble radius and U the terminal velocity, and this agrees with the result of Levich (1949) who argued from the viscous dissipation in the potential flow round the bubble. The next term in an expansion of the drag in descending fractional powers of R is found and the results compared with experiment.

1. Introduction

In a recent paper (Moore 1959) the author proposed a model of the flow past a rising spherical gas bubble at large Reynolds numbers R. The flow was everywhere only slightly disturbed from the irrotational flow past a sphere, and there was a thin boundary layer at the bubble surface in which the perturbation to the basic irrotational flow varied rapidly to allow the stress to change from its value $O(R^{-1})$ in the irrotational flow to the value zero required by the physical conditions at the bubble surface. It was further suggested that the drag force on the bubble could be calculated from the normal viscous stress of the irrotational flow and the result for the drag coefficient

$$C_D = 32/R \tag{1.1}$$

was obtained, where

$$R = 2aU\rho/\mu, \tag{1.2}$$

where the bubble of radius a rises with velocity U, ρ is the density and μ the viscosity of the liquid surrounding the bubble. However, soon after the paper was published, Dr G. K. Batchelor pointed out to the author that if the drag was calculated from the total viscous dissipation in the region of irrotational flow, a different result

$$C_D = 48/R \tag{1.3}$$

is achieved and suggested that the discrepancy was due to the neglect of pressure forces in the boundary layer. It was subsequently discovered that the result (1.3) had been obtained by the dissipation method by Levich (1949) and by Ackeret (1952).

In the present paper an examination of the boundary-layer equations is undertaken with a view of deciding between (1.1) and (1.3). In §2 Batchelor's suggestion is shown to be correct, since it is found that the pressure forces are

$O(R^{-1})$ and so make a significant contribution to C_D. The boundary-layer equations are linear and are solved analytically. The perturbation to the velocity field is $O(R^{-\frac{1}{2}})$ and the solution is well behaved except in the vicinity of the rear stagnation point, where the velocity and pressure fields have singularities.

The flow in the region of the rear stagnation point of the bubble cannot be described by a boundary-layer theory. An examination of the exact equations undertaken in §3 shows however that viscous forces are unimportant in this region so that the flow is governed by the inviscid equations. Moreover, the regions of validity of the boundary-layer equations and the inviscid approximation overlap and a matching procedure is possible. The determination of the flow in the rear stagnation region is simplified by the fact that the flow field is still only slightly disturbed from the potential flow.†

The vorticity created by the action of the viscous stresses in the boundary layer is convected downstream of the bubble to form a 'wake' of breadth $O(R^{-\frac{1}{4}})$. The perturbation velocity in this region is, however, only $O(R^{-\frac{1}{2}})$ and it is clear that the wake would not be observable in practice. Thus it is important to see whether the predicted absence of a wake is confirmed by observation. Unfortunately no clear experimental evidence exists, but a discussion of that available is undertaken in §5.

§4 is concerned with the drag force experienced by the bubble. Since an analytic solution for the rear stagnation region has not been found the drag cannot be calculated directly but must be found by a momentum argument. Agreement with Levich's result is achieved and it is further shown, by extending the dissipation calculation to include the contribution from the boundary layer and wake, that

$$C_D = \frac{48}{R}\left\{1 - \frac{2{\cdot}2\ldots}{R^{\frac{1}{2}}} + O(R^{-\frac{5}{6}})\right\}. \qquad (1.4)$$

The comparison of (1.4) with experiment is hampered by the onset of sensible distortion of the bubble from the spherical at Reynolds numbers of about 100, at which values the second term in (1.4) is not small compared to the first. Now the shape of the bubble is determined by the pressure field of the basic irrotational flow (Moore 1959) so that the amount of distortion is determined by the quantity $\rho U^2 a/T$, where T is the surface tension. Levich's result shows that $U \propto \rho g a^2/\mu$ so that the distortion is proportional to $R^{\frac{5}{3}} M^{\frac{1}{3}}$, where $M = g\mu^4/\rho T^3$ is a dimensionless group, introduced by Haberman & Morton (1953), which depends only on the liquid properties. The value of R at which, say, 10% distortion will occur is thus a function of the liquid properties (for bubbles in water it is about 200) and is a decreasing function of M. In §5 a comparison of theory and experiment is attempted for the two liquids with smallest M for which data are available. The agreement is fair and suggests that (1.4) represents some improvement on Levich's original result.

A recent analysis of the problem by Chao (1962), which appeared whilst the work presented here was being written up, takes a view conflicting with the present analysis. Chao investigates the boundary layer and arrives at an equation for the surface component of the velocity in agreement with the author's (2.28)

† That this might be the case was suggested to the author by Dr G. K. Batchelor.

derived below, but the neglect of curvature terms in the equation for the radial component leads to the conclusion that pressure forces are unimportant in the boundary layer. Thus incorrect results in agreement with the author's earlier theory are obtained.

2. The boundary layer

Adopt spherical polar co-ordinates at the centre of the sphere with the axis OZ pointing upsteam parallel to the undisturbed velocity U at infinity. Then if q'_r and q'_θ are the velocity components in this system the Navier–Stokes equations and the equation of continuity are

$$q'_\theta \frac{\partial q'_\theta}{r\,\partial\theta} + q'_r \frac{\partial q'_\theta}{\partial r} + \frac{q'_r q'_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p'}{\partial\theta} + \nu V'_\theta, \tag{2.1}$$

$$q'_\theta \frac{\partial q'_r}{r\,\partial\theta} + q'_r \frac{\partial q'_r}{\partial r} - \frac{q'^2_\theta}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} + \nu V'_r, \tag{2.2}$$

$$\frac{\partial q'_r}{\partial r} + \frac{2q'_r}{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(q'_\theta \sin\theta) = 0, \tag{2.3}$$

where $\nu V'_\theta$, $\nu V'_r$ represent the viscous forces, and

$$V'_\theta = \nabla^2 q'_\theta - \frac{q'_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial q'_r}{\partial\theta} \tag{2.4}$$

and

$$V'_r = \nabla^2 q'_r - \frac{2q'_r}{r^2} - \frac{2\cot\theta}{r^2} q'_\theta - \frac{2}{r^2}\frac{\partial q'_\theta}{\partial\theta}. \tag{2.5}$$

It is clear that (2.1), (2.2) and (2.3) are satisfied by the irrotational velocity field

$$\bar{q}_\theta = U\sin\theta\left(1 + \frac{a^3}{2r^3}\right), \tag{2.6}$$

$$\bar{q}_r = -U\cos\theta\left(1 - \frac{a^3}{r^3}\right), \tag{2.7}$$

and this velocity field satisfied the conditions at infinity and the condition that the normal velocity component should vanish on the sphere. However (2.6) and (2.7) do not give a vanishing tangential stress $p_{r\theta}$ at the bubble surface and, as argued by Moore (1959), this will lead at large Reynolds numbers to a boundary layer at the bubble surface in which the stress falls to zero.

It is convenient to write

$$q'_\theta = \bar{q}_\theta + q_\theta, \tag{2.8}$$

$$q'_r = \bar{q}_r + q_r, \tag{2.9}$$

$$p' = \bar{p} + p, \tag{2.10}$$

and one finds that q_θ, q_r satisfy the equations

$$\bar{q}_\theta \frac{\partial q_\theta}{r\,\partial\theta} + q_\theta \frac{\partial \bar{q}_\theta}{r\,\partial\theta} + \bar{q}_r \frac{\partial q_\theta}{\partial r} + q_r \frac{\partial \bar{q}_\theta}{\partial r} + \frac{\bar{q}_r q_\theta}{r} + \frac{q_r \bar{q}_\theta}{r} + \frac{q_r q_\theta}{r} + q_\theta \frac{\partial q_\theta}{r\,\partial\theta} + q_r \frac{\partial q_\theta}{\partial r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu V_\theta, \tag{2.11}$$

$$\bar{q}_\theta \frac{\partial q_r}{r\,\partial\theta} + q_\theta \frac{\partial \bar{q}_r}{r\,\partial\theta} + \bar{q}_r \frac{\partial q_r}{\partial r} + q_r \frac{\partial \bar{q}_r}{\partial r} - 2\frac{q_\theta \bar{q}_\theta}{r} + q_\theta \frac{\partial q_r}{r\,\partial\theta} + q_r \frac{\partial q_r}{\partial r} - \frac{q_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu V_r, \tag{2.12}$$

$$\frac{\partial q_r}{\partial r} + \frac{2q_r}{r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(q_\theta\sin\theta) = 0. \tag{2.13}$$

The boundary conditions to be satisfied are

$$q_r, q_\theta \to 0 \quad \text{as} \quad r \to \infty, \tag{2.14}$$

$$q_r = 0 \quad \text{on} \quad r = a, \tag{2.15}$$

and
$$p_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial}{\partial\theta}(\bar{q}_r + q_r) + r\frac{\partial}{\partial r}\left(\frac{\bar{q}_\theta}{r} + \frac{q_\theta}{r}\right)\right] = 0 \quad \text{on} \quad r = a. \tag{2.16}$$

If the boundary-layer thickness is of order δ then, symbolically, one has

$$\frac{\partial}{\partial r} = O\left(\frac{1}{\delta}\right), \tag{2.17}$$

where the operator is applied to q_θ and q_r, and (2.16) becomes

$$\frac{\partial q_\theta}{\partial r} = -r\frac{\partial}{\partial r}\left(\frac{\bar{q}_\theta}{r}\right)\bigg|_{r=a} = +\frac{3U}{a}\sin\theta. \tag{2.18}$$

Now the left-hand side of (2.18) is $O(1)$ so that this equation shows that $q_\theta = O(\delta)$ and hence, from the continuity equation (2.13), $q_r = O(\delta^2)$. In the boundary-layer region one also has $\bar{q}_\theta = O(1)$, $\bar{q}_r = O(\delta)$ (since $\bar{q}_r = 0$ on $r = a$) and $\partial\bar{q}_r/\partial r = O(1)$. Thus the orders of magnitude of the various terms in (2.11) and (2.12) are

$$\delta + \delta + \delta + \delta^2 + \delta^2 + \delta^2 + \delta^3 + \delta^2 + \delta^2 = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \frac{\nu}{\delta}, \tag{2.19}$$

$$\delta^2 + \delta^2 + \delta^2 + \delta^2 + \delta + \delta^3 + \delta^3 + \delta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu. \tag{2.20}$$

It follows from (2.19) that viscous and inertia forces will balance if

$$\delta^2 = O(\nu). \tag{2.21}$$

The order of magnitude of the pressure term in (2.19) must now be determined. If $q_\theta \to 0$ exponentially rapidly as the outer edge of the boundary layer is approached one sees from (2.11) that the pressure variations satisfy the equation

$$q_r\left(\frac{\partial\bar{q}_\theta}{\partial r} + \frac{\bar{q}_\theta}{r}\right) = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}. \tag{2.22}$$

It follows from (2.6) that the term in brackets is $O(\delta)$ in the boundary layer so that, since $q_r = O(\delta^2)$, $p = O(\delta^3)$ at the outer edge of the boundary layer. Furthermore, (2.20) shows that $\partial p/\partial r = O(\delta)$ so that, in view of the above argument, one has

$$p = O(\delta^2) \tag{2.23}$$

in the boundary layer. Thus, as pointed out by Batchelor privately, p contributes to the drag on the bubble to the same order as the viscous stresses—a fact which was overlooked by the author (Moore 1959).

If one retains only the terms of $O(\delta)$ on the left-hand side of (2.11) one has

$$\bar{q}_\theta\frac{\partial q_\theta}{r\partial\theta} + q_\theta\frac{\partial\bar{q}_\theta}{r\partial\theta} + \bar{q}_r\frac{\partial q_\theta}{\partial r} = \nu\frac{\partial^2 q_\theta}{\partial r^2}. \tag{2.24}$$

Let
$$q_\theta = U\delta u, \tag{2.25}$$
$$r - a = a\delta y, \tag{2.26}$$
$$\delta^2 = \frac{\nu}{aU} = \frac{2}{R}, \tag{2.27}$$
where R is the Reynolds number of the flow; then, on substituting the values assumed by $\bar{q}_\theta$, $\bar{q}_r$ in the boundary layer, (2.24) becomes
$$\frac{3}{2}\frac{\partial}{\partial\theta}(u\sin\theta) - 3y\cos\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2},\dagger \tag{2.28}$$
with the boundary conditions
$$u = 0 \quad \text{where} \quad \theta = 0 \text{ for all } y, \tag{2.29}$$
$$u \to 0 \quad \text{as} \quad y \to \infty, \quad 0 \leqslant \theta \leqslant \pi, \tag{2.30}$$
$$\partial u/\partial y = 3\sin\theta \quad \text{on} \quad y = 0, \quad 0 \leqslant \theta \leqslant \pi, \tag{2.31}$$
where (2.31) follows directly from (2.18).

The solution of (2.28) can be found by standard integral transform methods. We find
$$u = -6\sin\theta\,\chi^{\frac{1}{2}} f(y/2\chi^{\frac{1}{2}}), \tag{2.32}$$
where
$$f(t) = \pi^{-\frac{1}{2}}\exp(-t^2) - t\,\mathrm{erfc}\,t, \tag{2.33}$$
and
$$\chi(\theta) = \tfrac{2}{3}\operatorname{cosec}^4\theta\,(\tfrac{2}{3} - \cos\theta + \tfrac{1}{3}\cos^3\theta). \tag{2.34}$$
Clearly $\chi^{\frac{1}{2}}$ plays the role of the boundary-layer thickness. One may show that $\chi \to \frac{1}{6}$ as $\theta \to 0$ and that $\chi > 0$ for $0 \leqslant \theta \leqslant \pi$ so that the boundary layer starts smoothly at $\theta = 0$ and is well behaved for $0 \leqslant \theta < \pi$. Near $\theta = 0$ the boundary layer has a constant thickness and this resembles the boundary layer near the forward stagnation point of a solid cylinder. The nature of the solution near $\theta = \pi$ will be discussed below.

The pressure and radial component of velocity may now be determined. Retaining the most significant terms in (2.21) yields
$$\frac{\partial p}{\partial y} = 3\rho U^2\delta^2\sin\theta\,u(y,\theta), \tag{2.35}$$
so that, since $p = O(\delta^3)$ as $y \to \infty$,
$$p = -3\rho U^2\delta^2\sin\theta\int_y^\infty u(y',\theta)\,dy'. \tag{2.36}$$
Substituting for u in this integral one finds that
$$p = 36\rho U^2\delta^2\sin\theta\,\chi(\theta)\int_{y/2\chi^{\frac{1}{2}}}^\infty f(t)\,dt. \tag{2.37}$$

If one writes $q_r = U\delta^2 v$, the equation of continuity (2.13) and the boundary condition $v = 0$ at $y = 0$ show that
$$v = \frac{2\chi^{\frac{1}{2}}}{\sin\theta}\left\{h'(\theta)\int_0^{y/2\chi^{\frac{1}{2}}} f(t)\,dt + h(\theta)\left[2\cot\theta - \frac{1}{3\chi\sin\theta}\right]\int_0^{y/2\chi^{\frac{1}{2}}} tf'(t)\,dt\right\}, \tag{2.38}$$
where
$$h(\theta) = 6\sin^2\theta\,\chi^{\frac{1}{2}}. \tag{2.39}$$
This completes the determination of the boundary-layer solution.

† This is identical with Chao's boundary-layer equation, and the solutions are in agreement, allowing for an error in Chao's boundary condition.

3. The nature of the flow near the rear stagnation point

It is easy to verify that the solution for u, v and p given in §2 is well behaved except in the neighbourhood of $\theta = \pi$. Now

$$\chi \sim \frac{8}{9(\pi-\theta)^4} \quad \text{as} \quad \theta \to \pi, \tag{3.1}$$

so that, on writing $\phi = \pi - \theta$, we have

$$u(y,\theta) \sim -\frac{3}{\phi}\left\{\frac{4\sqrt{2}}{3\sqrt{\pi}}\exp\left(-\tfrac{9}{32}y^2\phi^4\right) - y\phi^2\operatorname{erfc}\left(\frac{3y\phi^2}{4\sqrt{2}}\right)\right\}. \tag{3.2}$$

Thus $q_\theta \sim O(\delta/\phi)$ as $\phi \to 0$, so that the boundary-layer solution is invalid for small ϕ. This is to be expected, since the flow must turn sharply as it approaches the rear axis of symmetry and the assumption that $\partial/\partial\theta \ll \partial/\partial r$ involved in the boundary-layer approximation will break down.

As a first step towards determining the nature of the flow in this region it is profitable to re-examine the orders of magnitude, according to the boundary-layer approximation, of the terms in the exact equations of motion (2.11) and (2.12). The solutions obtained for p and v show that $p = O(\delta^2/\phi^2)$, $q_r = O(\delta^2/\phi^4)$ as $\phi \to 0$; furthermore, since $\chi^{\frac{1}{2}}(\theta)$ plays the role of the boundary-layer thickness, $\partial/\partial r = O(\phi^2/\delta)$ when applied to q_r, q_θ and p. Using these estimates and noticing that $\bar{q}_\theta = O(\phi)$, $\bar{q}_r = O(\delta/\phi^2)$ in the region in question, one has for the orders of the terms in (2.11) and (2.12),

$$\underline{\frac{\delta}{\phi}} + \underline{\frac{\delta}{\phi}} + \underline{\frac{\delta}{\phi}} + \frac{\delta^2}{\phi^3} + \frac{\delta^2}{\phi^3} + \frac{\delta^3}{\phi^5} + \frac{\delta^2}{\phi^3} + \frac{\delta^2}{\phi^3} = \frac{\delta^2}{\phi^3} + \nu\left(\underline{\frac{\phi^3}{\delta}} + \frac{\delta}{\phi^3} + \frac{\delta^2}{\phi^5}\right) \tag{3.3}$$

and

$$\frac{\delta^2}{\phi^4} + \frac{\delta^2}{\phi^2} + \frac{\delta^2}{\phi^4} + \frac{\delta^2}{\phi^4} + \underline{\delta} + \frac{\delta^3}{\phi^6} + \frac{\delta^3}{\phi^6} + \frac{\delta^2}{\phi^2} = \underline{\delta} + \nu\left(\frac{1}{\phi^2} + \frac{\delta^2}{\phi} + \frac{\delta}{\phi^2}\right). \tag{3.4}$$

The terms underlined are those retained in the boundary-layer approximation of §2 and it can be seen that the retained inertia terms are larger than the neglected inertia terms in both (3.3) and (3.4) so long as $\phi \gg \delta^{\frac{1}{4}}$. Furthermore, if $\delta^{\frac{1}{4}} \ll \phi \ll 1$ the viscous terms are negligible compared to the inertia terms, since the viscous terms behave like $\delta\phi^3$ in (3.3) and are $O(\delta^2/\phi^2)$ in (3.4) in comparison with inertia terms $O(\delta/\phi)$ and $O(\delta)$ respectively.

These estimates show that there is a range of values of ϕ, namely $\delta^{\frac{1}{4}} \ll \phi \ll 1$, in which the perturbation to the potential flow is still confined to a thin boundary layer at the bubble surface and is still given correctly by (3.2), but in which viscous forces are negligible. For smaller values of ϕ the assumptions underlying the boundary-layer equations will fail. However, in view of the inertial character of the dynamics in the final stages of the boundary layer, it is plausible to suppose that viscous forces will remain unimportant in determining the perturbation throughout the region near the rear stagnation point. Thus it will be assumed that very near the rear stagnation point the relevant approximation is that of a small departure from the potential flow $(\bar{q}_\theta, \bar{q}_r)$ where the departure is governed by the full *inviscid* equations, no restrictions being placed on the relative orders of magnitude of $\partial/\partial r$ and $\partial/\partial\theta$. Moreover, since the flow is assumed inviscid if $1 \gg \phi$ and the boundary-layer approximation is valid if $\phi \gg \delta^{\frac{1}{4}}$, the boundary-layer approximation and the inviscid approximation have a common region of validity and a matching procedure may be envisaged. The assumption that the flow in the stagnation region is inviscid will be examined *a posteriori*.

Now in an axisymmetric steady inviscid flow the dynamics may be summarized by the two equations

$$\omega/m = B(\psi) \tag{3.5}$$

and

$$p + \tfrac{1}{2}\rho\mathbf{u}^2 + C(\psi) = 0, \tag{3.6}$$

where ω is the azimuthal component of the vorticity, m the distance from the axis of symmetry, and ψ the Stokes stream function and where $B(\psi)$ and $C(\psi)$ are functions to be determined. In general these equations are not very fruitful, since the real problem is to determine ψ. However, in the present case, we can determine the vorticity ω of the perturbation flow (q_θ, q_r) to the first order by taking ψ to be the known stream function $\overline{\psi}$ of the basic potential flow $(\overline{q}_\theta, \overline{q}_r)$. Thus in (3.5) we replace $B(\psi)$ by $B(\overline{\psi})$. In physical terms, the approximation amounts to assuming that the distortion of the potential flow by the perturbation has only a second-order effect on the convection of the vorticity created in the boundary layer, so that this vorticity is, in effect, convected passively.

The function $B(\overline{\psi})$ will be determined by insisting that (3.5) yields the correct vorticity in the overlap region $\delta^{\frac{1}{4}} \ll \phi \ll 1$. The vorticity in the boundary layer is $-\partial q_\theta/\partial r$ and from (3.2) one has

$$\frac{\partial q_\theta}{\partial r} = \frac{3U}{a}\phi\,\mathrm{erfc}\left(\frac{3y\phi^2}{4\sqrt{2}}\right). \tag{3.7}$$

Thus, bearing in mind that, to the order of the approximation,

$$m = a\phi, \tag{3.8}$$

one has

$$\frac{\omega}{m} = -\frac{3U}{a^2}\,\mathrm{erfc}\left(\frac{3y\phi^2}{4\sqrt{2}}\right). \tag{3.9}$$

The stream function of the potential flow is

$$\overline{\psi} = \tfrac{1}{2}U\sin^2\theta\left(r^2 - \frac{a^3}{r}\right), \tag{3.10}$$

so that in the rear stagnation region

$$\overline{\psi} = \tfrac{3}{2}U\delta y a^2\phi^2. \tag{3.11}$$

Equations (3.11) and (3.9) together show that in the final stage of the boundary layer ω/m is, indeed, constant on streamlines of the potential flow and they further imply that

$$B(\overline{\psi}) = -\frac{3U}{a^2}\,\mathrm{erfc}\left(\frac{\overline{\psi}}{2\sqrt{2}\,Ua^2\delta}\right). \tag{3.12}$$

The function $C(\overline{\psi})$ may also be determined by matching. Since Bernoulli's equation applies in the overlap region,

$$p + \tfrac{1}{2}\rho[(\overline{q}_\theta + q_\theta)^2 + (\overline{q}_r + q_r)^2] + C(\overline{\psi}) = 0, \tag{3.13}$$

and q_r, q_θ are known from the boundary-layer solution. Thus

$$C(\overline{\psi}) = -\rho \bar{q}_\theta q_\theta + O(\delta^2), \tag{3.14}$$

so that, on using the explicit solution (3.2) for q_θ in the final stage of the boundary layer, one has

$$C(\overline{\psi}) = \rho U^2 \delta \left[\frac{6\sqrt{2}}{\sqrt{\pi}} \exp\left(-\frac{\overline{\psi}^2}{8U^2\delta^2 a^4}\right) - \frac{3\overline{\psi}}{U\delta a^2} \operatorname{erfc}\left(\frac{\overline{\psi}}{2\sqrt{2}\,U\delta a^2}\right)\right]. \tag{3.15}$$

Thus, since the pressure and vorticity are known at every point of the stagnation region, the details of the flow there may, in principle, be calculated. As a start, one may estimate the orders of magnitude of the relevant quantities.

The vorticity produced in the boundary layer will be confined to a region bounded by the sphere's surface and that stream surface of the potential flow which is at the outer edge of the boundary layer in the matching region. In the matching region this stream surface is characterized by

$$r-a = aO(\delta/\phi^2), \tag{3.16}$$

so that its equation is, by (3.10),

$$(r-a)\,\phi^2 = \delta G, \tag{3.17}$$

where G is a constant of order unity. Thus when the stream surface makes a finite angle with the bubble surface, so that $r-a = O(a\phi)$, both $r-a$ and $a\phi$ are $aO(\delta^{\frac{1}{3}})$. Thus the stagnation region is of size $O(R^{-\frac{1}{6}})$.

To determine the velocities of the perturbation one may remark that the vorticity is $O(R^{-\frac{1}{6}})$ and that the velocities must change significantly in distances of $O(R^{-\frac{1}{6}})$ so that the velocities are $O(R^{-\frac{1}{3}})$, which is smaller than the velocities of the potential flow by a factor $R^{-\frac{1}{3}}$ when R is large. Thus the streamlines are only slightly displaced by the perturbation, as assumed. Furthermore, the viscous forces are $O(R^{-1})$ whilst the inertia forces are $O(R^{-\frac{1}{6}})$ so that the neglect of viscous forces in the stagnation region is self-consistent.

To discuss the stagnation region in more detail one may introduce dimensionless co-ordinates z and s, where

$$r-a = a\delta^{\frac{1}{3}}z \tag{3.18}$$

and

$$\phi = s\delta^{\frac{1}{3}}. \tag{3.19}$$

The curvature of the bubble wall will be unimportant in the stagnation region and one has effectively a local system of cylindrical polar co-ordinates. It is convenient to represent the velocities in terms of a dimensionless stream function ψ such that

$$q_z = U\delta^{\frac{2}{3}} \frac{1}{s}\frac{\partial \psi}{\partial s}, \tag{3.20}$$

$$q_s = -U\delta^{\frac{2}{3}} \frac{1}{s}\frac{\partial \psi}{\partial z}; \tag{3.21}$$

and, since the azimuthal component of vorticity is $(\partial q_s/\partial z - \partial q_z/\partial s)\,a^{-1}\delta^{\frac{1}{3}}$, ψ must satisfy the equation

$$\frac{\partial^2 \psi}{\partial s^2} - \frac{1}{s}\frac{\partial \psi}{\partial s} + \frac{\partial^2 \psi}{\partial z^2} = 3s^2 \operatorname{erfc}\left(\frac{3zs^2}{4\sqrt{2}}\right). \tag{3.22}$$

ψ must also satisfy the boundary conditions

$$\psi = O(s^2) \quad \text{as} \quad s \to 0, \tag{3.23}$$

$$\partial\psi/\partial s = 0 \quad \text{on} \quad z = 0, \tag{3.24}$$

and the conditions

$$\frac{1}{s}\frac{\partial\psi}{\partial s}, \quad \frac{1}{s}\frac{\partial\psi}{\partial z} \to 0 \quad \text{as} \quad s^2+z^2 \to \infty, \tag{3.25}$$

since the perturbations are of a smaller order in R outside the stagnation region. The boundary condition on the tangential stress of the perturbation is satisfied identically by any solution of the above system.

Unfortunately the writer has not been able to solve the problem posed by equations (3.22) to (3.25) and thus the details of the flow in the stagnation region are unknown.

However, it seems reasonable to assume that a solution will exist and, fortunately, it proves possible to calculate the drag force experienced by the bubble without such detailed knowledge.

The vorticity created in the boundary layer will eventually be carried downstream of the bubble to form a 'wake' and one may determine the orders of magnitude of the perturbation to the potential flow in the wake region by an argument similar to that employed for the stagnation region. When the stream surface bounding the vorticity distribution is effectively cylindrical with generators parallel to the axis of symmetry, one has $r-a = O(a)$, so that (3.17) shows that $\phi = O(\delta^{\frac{1}{2}})$. Thus the wake is of breadth $O(R^{-\frac{1}{4}})$. The vorticity is thus also $O(R^{-\frac{1}{4}})$ so that the axial component of the perturbation velocity is $O(R^{-\frac{1}{2}})$. As has been mentioned in § 1, the present theory predicts that the wake will be unobservable in practice.

The details of the perturbation in the wake can be obtained without difficulty, since the streamlines of the potential flow are essentially parallel and the analogue of equation (3.22) is correspondingly simpler. Furthermore, the above estimates show that the viscous forces in the wake are $O(R^{-1})$ as compared to inertia forces of $O(R^{-\frac{1}{2}})$ Thus the diffusing of the vorticity created in the boundary layer and thereafter convected by the potential flow may be neglected, at least in the initial stages of the wake development. In fact, it will be shown to be negligible until distances downstream of the bubble of $O(aR^{\frac{1}{2}})$ are reached, so that the dynamics of the wake will be governed initially by equations (3.5) and (3.6), where the functions $B(\overline{\psi})$ and $C(\overline{\psi})$ are as determined by the matching procedure.

A further simplification becomes possible if one assumes, as is established below, that viscous forces are negligible at downstream distances less than $O(aR^{\frac{1}{2}})$. For then, since $R \gg 1$, there exists a range of distances d satisfying the inequality $a \ll d \ll aR^{\frac{1}{2}}$ at which the basic potential flow is essentially parallel and of constant velocity U and the wake is still subject to inviscid dynamics. Adopting cylindrical polar co-ordinates m and x where, as above, m is distance from the axis of symmetry and x is distance downstream from the bubble centre measured along the axis of symmetry, the stream function of the essentially parallel basic potential flow is

$$\overline{\psi} = \tfrac{1}{2}Um^2. \tag{3.26}$$

If q_m and q_x are the components of the perturbation velocity, the perturbation vorticity is $-\partial q_x/\partial m$ (the other contribution is negligible since x gradients are small) so that from (3.5)

$$-\frac{\partial q_x}{\partial m} = mB(\tfrac{1}{2}Um^2), \tag{3.27}$$

where the function B is as determined by the matching procedure. Moreover

$$q_x \to 0 \quad \text{as} \quad m \to \infty, \tag{3.28}$$

and using this boundary condition and the form for B given in (3.12) one has

$$q_x = -6\sqrt{2}\,U\delta\{\pi^{-\frac{1}{2}}\exp(-\sigma^2) - \sigma\operatorname{erfc}\sigma\}, \tag{3.29}$$

where σ is defined by

$$\sigma = \frac{m^2}{4\sqrt{2}\,a^2\delta}. \tag{3.30}$$

When viscous diffusion can no longer be ignored the perturbation q_x will be governed by the equation

$$U\frac{\partial q_x}{\partial x} = \nu\left(\frac{\partial^2 q_x}{\partial m^2} + \frac{1}{m}\frac{\partial q_x}{\partial m}\right), \tag{3.31}$$

and it is clear that, when the viscous and inertia terms balance,

$$U/L \sim \nu/a^2R^{-\frac{1}{2}}, \tag{3.32}$$

where L is the distance downstream; thus

$$L = O(aR^{\frac{1}{2}}). \tag{3.33}$$

Hence, as has been asserted, there exists a region intermediate to a and $aR^{\frac{1}{2}}$ in which, although the x derivative is too small to affect the velocity distribution significantly, the inertia term whose magnitude it controls is an order of magnitude larger than the viscous term.

It will be convenient at this point to summarize, in order of magnitude terms, the complete flow pattern suggested by the above considerations. There is a boundary layer of thickness $O(R^{-\frac{1}{2}})$ at the bubble surface in which the perturbation from the potential velocity field of (2.6) and (2.7) is $O(R^{-\frac{1}{2}})$. At the rear of the bubble there is a region of linear size $O(R^{-\frac{1}{6}})$ in which the vorticity from this boundary layer is transferred to the wake and in this region the departures from the potential flow are $O(R^{-\frac{1}{3}})$. This stagnation region feeds the vorticity produced in the boundary layer into the wake, which is of diameter $O(R^{-\frac{1}{4}})$ and in which the perturbation to the potential velocity field is $O(R^{-\frac{1}{2}})$. Outside this combined boundary layer, stagnation region and wake system there is an irrotational secondary flow determined by the outflow from the system. The major contribution seems to be from the stagnation region; velocities are $O(R^{-\frac{1}{3}})$ and the region has area $O(R^{-\frac{1}{3}})$ so that the secondary flow velocities are $O(R^{-\frac{2}{3}})$.

The appearance of regions of size $O(R^{-\frac{1}{6}})$ and associated velocity fields of $O(R^{-\frac{1}{3}})$ is surprising, but seems to be due to the three-dimensional nature of the flow. The powers of $\delta^{\frac{1}{3}}$ arise from the form of the stream function for three-dimensional stagnation flow and it may be seen that no such powers would arise in the corresponding solution for flow past a two-dimensional bubble.

4. The drag on the bubble

It is of interest to calculate the drag force experienced by the bubble. The obvious way to do this would be to integrate the normal stress over the bubble surface but unfortunately, as already stated in §3, the author has not found it possible to determine the perturbation velocities (q_θ, q_r) in the stagnation region in closed form. However, one may calculate the first approximation to the drag by a momentum argument.

Let S and S' be infinite plane surfaces normal to the axis of symmetry. S a large distance ahead of the bubble and S' a large distance behind. More precisely, one demands that the distance d downstream should satisfy $a \ll d \ll aR^{\frac{1}{2}}$ ensuring that the flow is sensibly parallel but that viscous modifications to the wake are unimportant. Then (Landau & Lifshitz 1959, p. 72) the drag D is given by

$$D = \int_S (p+\rho U q_x)\, dS - \int_{S'} (p+\rho U q_x)\, dS, \tag{4.1}$$

where q_x is the x component of the perturbation to the potential flow. This perturbation is determined from the outflow from the boundary layer and is thus $O(R^{-\frac{2}{3}})$ but its exact value will not be required. Now Bernoulli's equation gives for the perturbation pressure

$$p + \tfrac{1}{2}\rho \mathbf{U}^2 + \rho \mathbf{U}.\mathbf{u} + C(\overline{\psi}) = 0, \tag{4.2}$$

where $C(\overline{\psi})$ is zero except in the rear stagnation region, in the boundary layer, and in the wake.

Thus
$$p = -\rho U q_x - C(\overline{\psi}) + O(R^{-\frac{4}{3}}) \tag{4.3}$$

at large axial distances from the bubble so that, on substituting in (4.1), one has

$$D = \int_{S'} C(\overline{\psi})\, dS - \int_S C(\overline{\psi})\, dS. \tag{4.4}$$

Now $C(\overline{\psi})$ is zero at every point of S, but it is non-zero at those parts of S' inside the wake, so that

$$D = \int_{S'} C(\overline{\psi})\, dS. \tag{4.5}$$

In the region of uniform streaming

$$\overline{\psi} = \tfrac{1}{2} U m^2, \tag{4.6}$$

so that
$$D = \int_0^\infty \frac{2\pi}{U} C(\overline{\psi})\, d\overline{\psi}. \tag{4.7}$$

The function $C(\overline{\psi})$ has been determined above by the matching process and on inserting $C(\overline{\psi})$ in (4.7) and performing the integration one finds that

$$D = 12\pi \rho U^2 \delta^2 a^2, \tag{4.8}$$

so that, on using (2.27) to eliminate δ, one has

$$D = 12\pi\mu U a, \tag{4.9}$$

in agreement with Levich (1949).

Levich obtained his result by calculating the viscous dissipation in the potential flow and this suggests that, since the boundary-layer structure has been determined, one might improve upon his result by including the dissipation from the boundary layer, stagnation region and wake, provided that it can be shown that the contribution from the external secondary potential flow, which is unknown, is negligible.

The dissipation function is

$$\Phi' = \mu \frac{\partial u_i'}{\partial x_j}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right), \tag{4.10}$$

and since

$$u_i' = \bar{u}_i + u_i, \tag{4.11}$$

where $\bar{u}_i$ is the potential flow and u_i the perturbation to it, one has, exactly,

$$\Phi' = \bar{\Phi} + \Phi + \mu \frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu \frac{\partial u_i}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{4.12}$$

or

$$\Phi' = \bar{\Phi} + \Phi + 2\mu \frac{\partial \bar{u}_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + 2\mu \frac{\partial \bar{u}_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}. \tag{4.13}$$

This expression is next transformed so that the irrotational nature of $\bar{\mathbf{u}}$ can be used to simplify it. Thus

$$\Phi' = \bar{\Phi} + \Phi + 2\mu\left\{\frac{\partial}{\partial x_j}\left(u_i \frac{\partial \bar{u}_i}{\partial x_j}\right) - u_i \nabla^2 \bar{u}_i\right\} + 2\mu\left\{\frac{\partial}{\partial x_j}\left(u_i \frac{\partial \bar{u}_j}{\partial x_i}\right) - u_i \frac{\partial}{\partial x_i}\left(\frac{\partial \bar{u}_j}{\partial x_j}\right)\right\}, \tag{4.14}$$

and so, since $\bar{\mathbf{u}}$ is irrotational, one has

$$\Phi' = \bar{\Phi} + \Phi + 4\mu \frac{\partial}{\partial x_j}(u_i \bar{e}_{ij}), \tag{4.15}$$

where

$$\bar{e}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{4.16}$$

is the rate-of-strain tensor of the potential flow. Thus if D is the drag, one has

$$DU = \int_V \bar{\Phi}\, dV + \int_V \Phi\, dV + 4\mu \int_V \frac{\partial}{\partial x_j}(u_i \bar{e}_{ij})\, dV, \tag{4.17}$$

where V is the space exterior to the sphere. The divergence theorem applied to the last term yields

$$DU = \int_V \bar{\Phi}\, dV + \int_V \Phi\, dV - 4\mu \int_{S_1} n_j u_i \bar{e}_{ij}\, dS, \tag{4.18}$$

where $\mathbf{n}$ is the normal to the sphere's surface S_1, drawn into the fluid, and where the contributions from a large sphere at infinity have been assumed negligible in the transformation to a surface integral. We can also put

$$\int_V \Phi\, dV = \int_{\substack{\text{boundary}\\\text{layer}}} \Phi\, dV + \int_{\substack{\text{stagnation}\\\text{region}}} \Phi\, dV + \int_{\text{wake}} \Phi\, dV + \int_{\text{exterior}} \Phi\, dV, \tag{4.19}$$

where 'exterior' refers to the region of flow outside the boundary layer and wake and in which the flow is the basic irrotational flow plus an irrotational secondary flow determined by the outflow from the boundary layer and wake.

The various terms are now estimated. The velocity field of the secondary flow was shown in §3 to be $O(R^{-\frac{2}{3}})$ so that

$$\int_{\text{exterior}} \Phi \, dV = O(R^{-\frac{7}{3}}). \tag{4.20}$$

Using the estimates given in §3 for the stagnation region one has

$$\int_{\substack{\text{stagnation}\\ \text{region}}} \Phi \, dV = O(R^{-\frac{11}{6}}). \tag{4.21}$$

Thus one has

$$\int_V \Phi \, dV = \int_{\substack{\text{boundary}\\ \text{layer}}} \Phi \, dV + \int_{\text{wake}} \Phi \, dV + O(R^{-\frac{11}{6}}), \tag{4.22}$$

and since the first two integrals are $O(R^{-\frac{3}{2}})$ the other regions make no contribution to the second approximation to the dissipation. The surface integral receives a contribution $O(R^{-\frac{11}{6}})$ from that part of S in the stagnation region so that, finally,

$$DU = \int_V \bar{\Phi} \, dV + \int_{\substack{\text{boundary}\\ \text{layer}}} \Phi \, dV + \int_{\text{wake}} \Phi \, dV - 4\mu \int_{S_2} n_j u_i \bar{e}_{ij} \, dS, \tag{4.23}$$

where S_2 is the portion of the sphere's surface S_1 covered by the boundary layer. Now

$$\int_{\substack{\text{boundary}\\ \text{layer}}} \Phi \, dV = 2\pi a U^2 \mu \delta \int_{\theta=0}^{\pi} \int_{y=0}^{\infty} \left(\frac{\partial u}{\partial y}\right)^2 \sin\theta \, d\theta \, dy \tag{4.24}$$

and

$$4\mu \int_{S_2} q_\theta \bar{e}_{r\theta} \, dS = 8\pi a U^2 \mu \delta \int_0^{\pi} u(0,\theta)\,(-\tfrac{3}{2}\sin\theta)\sin\theta \, d\theta. \tag{4.25}$$

These integrals can be evaluated by inserting the solution for $u(y,\theta)$ given in §2.

The dissipation in the wake is slightly more troublesome to calculate. One has

$$\int_{\text{wake}} \Phi \, dV = \mu \int_{x=0}^{\infty} \int_{m=0}^{\infty} 2\pi m \left(\frac{\partial q_x}{\partial m}\right)^2 dm \, dx, \tag{4.26}$$

where, since the wake is effectively of length $aR^{\frac{1}{2}}$, $x = 0$ can be taken to be a position where the flow is parallel. The integral can be transformed by partial integration to yield

$$\int_{\text{wake}} \Phi \, dV = 2\pi\mu \int_{x=0}^{\infty} dx \left\{ \left[m q_x \frac{\partial q_x}{\partial m} \right]_0^{\infty} - \int_0^{\infty} q_x \frac{\partial}{\partial m}\left(m \frac{\partial q_x}{\partial m} \right) dm \right\}, \tag{4.27}$$

so that, on using the equation of motion (3.31) satisfied by q_x,

$$\int_{\text{wake}} \Phi \, dV = -2\pi\rho U \int_{x=0}^{\infty} \int_{m=0}^{\infty} q_x m \frac{\partial q_x}{\partial x} \, dm \, dx, \tag{4.28}$$

and on carrying out the x integration

$$\int_{\text{wake}} \Phi \, dV = \pi\rho U \int_{m=0}^{\infty} m \{ q_x^2|_{x=0} - q_x^2|_{x=\infty} \} \, dm. \tag{4.29}$$

From the familiar theory of the asymptotic structure of the laminar wake

$$q_x \sim \frac{A}{x} \exp\left\{-\frac{m^2 U}{4\nu x}\right\} \quad \text{as} \quad x \to \infty, \tag{4.30}$$

so that the contribution from $x = \infty$ can be estimated and is readily found to be zero. Thus

$$\int_{\text{wake}} \Phi \, dV = \pi \rho U \int_{m=0}^{\infty} m q_x^2|_{x=0} \, dm, \tag{4.31}$$

and on inserting the expression for q_x found in §3 one obtains

$$\int_{\text{wake}} \Phi \, dV = 144 \pi U^2 \mu a \delta \sqrt{2} \int_0^{\infty} \{\pi^{-\frac{1}{2}} \exp(-\sigma^2) - \sigma \operatorname{erfc} \sigma\}^2 \, d\sigma. \tag{4.32}$$

The definite integrals involved in (4.24), (4.25) and (4.32) can be evaluated in closed form by some lengthy manipulations and one finds that

$$D = 12\pi U a \mu \left\{1 - \frac{4\sqrt{2}\,(6\sqrt{3} + 5\sqrt{2} - 14)}{5\sqrt{\pi}\, R^{\frac{1}{2}}} + O(R^{-\frac{5}{6}})\right\} \tag{4.33}$$

or, in terms of the drag coefficient,

$$C_D = \frac{48}{R}\left(1 - \frac{2{\cdot}2\ldots}{R^{\frac{1}{2}}} + O(R^{-\frac{5}{6}})\right). \tag{4.34}$$

5. Comparison with experiment

The theory presented above predicts that there will be no observable wake behind a steadily rising gas bubble and this suggests comparison with the observations of Hartunian & Sears (1957). These workers observed a closed wake consisting of a single vortex ring behind a steadily rising bubble, but remarked that: 'It is significant that the wake could be seen only in the impure and more viscous liquids.' For larger bubbles oscillations occurred which are reminiscent of the oscillations of a freely falling solid and which suggest wake instability. Significantly, when bubbles of this critical size were released in distilled water they oscillated until the dye used to visualize the flow was shed, from which point on the bubbles rose steadily. Now it is well known that small bubbles in impure liquids behave as if the bubble surface were solid and Hartunian & Sears suggest that the dye contaminated the surface in their experiment. The cessation of oscillations once the dye is shed shows that a change in the structure of the wake occurs when the bubble surface becomes stress free, although whether or not the wake disappears is not established.

A more direct comparison with experiment is achieved if the theory is used to predict the velocities of steady rise of bubbles. Unfortunately, it is clear that the range of Reynolds numbers for which the predictions of §4 are applicable is very limited since one requires both that $R \gg 1$ and $R^{\frac{5}{3}} M^{\frac{1}{3}} \ll 1$. Furthermore, the experimental drag coefficients of Haberman & Morton (1953) are presented graphically on too small a scale to permit accurate comparison for this limited range of Reynolds numbers. With these facts in mind, (4.34) was used to compute velocities of rise for the two pure liquids, Varsol ($M = 4{\cdot}3 \times 10^{-10}$) and methyl

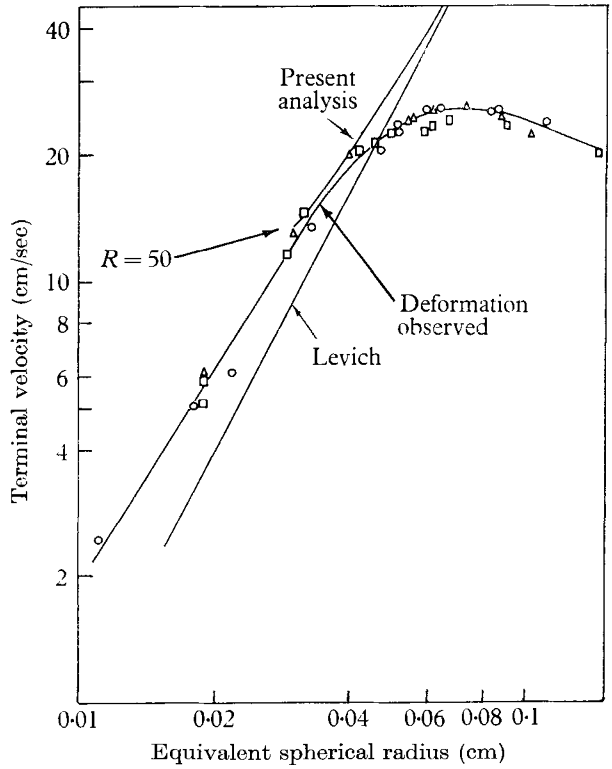

FIGURE 1. Comparison of velocity of rise computed from (4.34) (upper curve) and from Levich's result (lower curve) compared with Haberman & Morton's (1953) experimental curve for Varsol ($M = 4{\cdot}3 \times 10^{-10}$). The point at which deformation was first observed is approximately indicated.

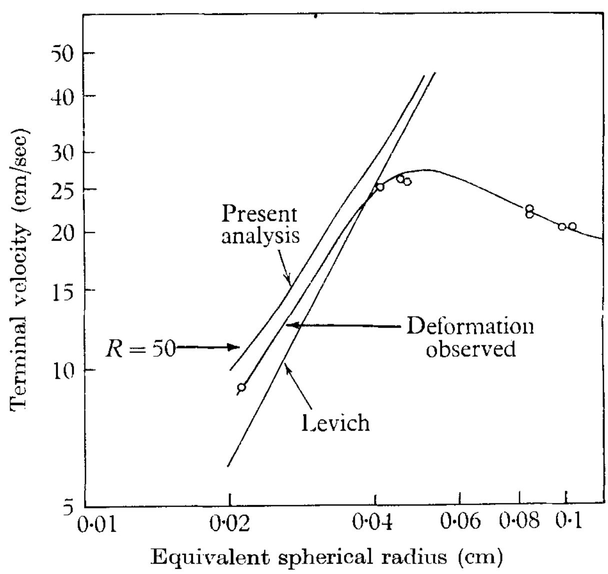

FIGURE 2. Comparison of velocity of rise computed from (4.34) (upper curve) and from Levich's result (lower curve) compared with Haberman & Morton's (1953) experimental curve for methyl alcohol ($M = 0{\cdot}89 \times 10^{-10}$). The point at which deformation was first observed is approximately indicated.

alcohol ($M = 0{\cdot}89 \times 10^{-10}$), with the smallest values of M amongst the liquids studied by Haberman & Morton. These workers give curves of the velocity of rise as a function of radius and the comparison is shown in figures 1 and 2.

It appears from this comparison that (4.34) is closer to the experimental curve than Levich's original result, though the theoretical and experimental curves diverge sharply at the larger velocities of rise. The point at which bubble distortion was first observed by Haberman & Morton is indicated and it seems likely that the divergence is due to deformation effects. The author hopes to make the effect of distortion on the velocity of rise the subject of a future paper.

The author has benefited from valuable discussions with Prof. A. Acrivos and Prof. L. Howarth and is particularly in debt to Dr G. K. Batchelor for his continued interest and advice.

REFERENCES

Ackeret, J. 1952 *Z. angew. Math. Phys.* **3**, 259.

Chao, B. T. 1962 *Phys. Fluids*, **5**, 69.

Haberman, W. L. & Morton, R. K. 1953 *David Taylor Model Basin Rep.* no. 802.

Hartunian, R. A. & Sears, W. R. 1957 *J. Fluid Mech.* **3**, 27.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics*. London: Pergamon Press.

Levich, V. 1949 *Zh. Eksptl. i Teoret. Fiz.* **19**, 18.

Moore, D. W. 1959 *J. Fluid Mech.* **6**, 113.